APPENDIX C

FLEXING OF LENGTH BARS

C.1 FLEXING OF A LENGTH BAR DUE TO ITS OWN WEIGHT

Any object lying in a horizontal plane will sag under its own weight unless it is infinitely stiff or is supported at many points along its length. For length bars this causes two problems. Firstly, if there is any sagging in the vicinity of the ends of the bar, this will cause the two end faces to tilt with respect to one another causing a bar with otherwise parallel faces to appear out of parallel. Secondly, since the material of the bar no longer lies in a straight line between the two end faces, the extra bending may cause the length of the bar, measured as the separation between the end faces, to become shorter than in its free state.

One solution is to measure the bars vertically, though this is not possible because of three reasons. Firstly, the relevant standards state that the bars should be measured in a horizontal plane, supported at two points termed the “Airy points” (see later), since this is how they will be used in practice. Secondly, a bar standing vertically will contract under its own weight, see Appendix D. Thirdly, the variation of refractive index between the top and bottom of the bar due to (i) the air pressure gradient due to the Earth’s gravitational field and (ii) the variation in the air temperature, contributes a significant measurement uncertainty.

Historical solutions such as floating the bar in mercury or supporting it on a system of 8 rollers or supports [1] have been rejected as hazardous or impractical. They also do not conform to the relevant standards. The chosen solution is to support the bar on two points whose positions are chosen to make the ends of the bar vertical and parallel with each other. These are termed the “Airy points” of the bar and their positions are usually engraved on the bar’s surface. The position of these points will now be derived.

C.2 DERIVATION OF POSITIONS OF AIRY POINTS

Consider a uniform solid bar of length L, cross-sectional moment of inertia I, and total weight W. This bar is supported at 2 points, symmetrically placed about its middle, separated by a distance S. Let the reactions at the two supports be R_1& R_2 as shown in figure C.1.

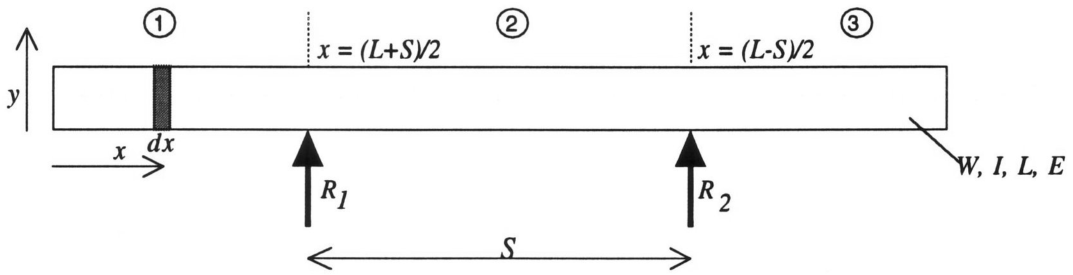

Figure C.1 *- Bar supported at two points*

Resolving vertically,

$$R_1 + R_2 = W, \quad R_1 = R_2$$
$$\therefore R_1 = R_2 = \frac{W}{2}$$

Now, split the bar into three sections (1) to (3) as shown in figure C.1, for the following analysis. In each section,

$$\sum bending\ moments = EI\frac{d^2y}{dx^2} \text{ (Bernoulli-Euler theory)}$$

Since the bar is uniform, EI is a constant, and as such will be removed from the following equations for simplicity.

In section (1)

$$\frac{d^2y}{dx^2} = \frac{Wx^2}{2L} \tag{C.1}$$

In section (2)

$$\frac{d^2y}{dx^2} = \frac{Wx^2}{2L} - R_1\left[x - \frac{(L-S)}{2}\right] \tag{C.2}$$

In section (3)

$$\frac{d^2y}{dx^2} = \frac{Wx^2}{2L} - R_1\left[x - \frac{(L-S)}{2}\right] - R_2\left[x - \frac{(L+S)}{2}\right] \tag{C.3}$$

Integrating equations (C.1) (C.2) and (C.3) gives, respectively,

$$\frac{dy}{dx} = \frac{Wx^3}{6L} + C_1 \tag{C.4}$$

$$\frac{dy}{dx} = \frac{Wx^3}{6L} - R_1\left[\frac{x^2}{2} - \frac{(L-S)x}{2}\right] + C_2 \tag{C.5}$$

$$\frac{dy}{dx} = \frac{Wx^3}{6L} - R_1\left[\frac{x^2}{2} - \frac{(L-S)x}{2}\right] - R_2\left[\frac{x^2}{2} - \frac{(L+S)x}{2}\right] + C_3 \tag{C.6}$$

The slope of the bar, $\frac{dy}{dx}$ must be continuous at the supports therefore equating (C.4) and (C.5), and substituting $x = \frac{L-S}{2}$ gives

$$C_1 = C_2 - R_1\left[\frac{(L-S)^2}{8} - \frac{(L-S)^2}{4}\right]$$

$$ie \quad C_1 = C_2 + R_1\left[\frac{(L-S)^2}{8}\right] \tag{C.7}$$

One constraint is that we require vertical end faces, *ie* $\left.\frac{dy}{dx}\right|_{x=0} = 0$

This implies that $C_1 = 0$ Substituting this result into (C.7) and using the fact that $R_1 = \frac{W}{2}$ gives

$$C_2 = -\frac{W}{2}\left[\frac{(L-S)^2}{8}\right] \tag{C.8}$$

Now, matching $\frac{dy}{dx}$ at $x = L$ gives

$$C_3 = -\frac{WL^2}{6} \tag{C.9}$$

With C_1, C_2, C_3 determined, equations (C.3) (C.4) and (C.5) completely describe the bending of the bar, once S is known. To find S, $\frac{dy}{dx}$ is matched at the boundary between regions (2) and (3).

In region (2)

$$\left.\frac{dy}{dx}\right|_{x=\frac{L+S}{2}} = \frac{W(L+S)^3}{48L} - \frac{W}{2}\left[\frac{(L+S)^2}{8} - \frac{(L-S)(L+S)}{4} + \frac{(L-S)^2}{8}\right] \tag{C.10}$$

and in region (3)

$$\left.\frac{dy}{dx}\right|_{x=\frac{L+S}{2}} = \frac{W(L+S)^3}{48L} - \frac{W}{2}\left[\frac{(L+S)^2}{8} - \frac{(L-S)(L+S)}{4} + \frac{(L+S)^2}{8} - \frac{(L+S)^2}{4}\right] - \frac{WL^2}{6}$$

(C.11)

Equating (10) and (11) gives

$$-\frac{W(L-S)^2}{16} = -\frac{W}{2}\left[-\frac{(L+S)^2}{8}\right] - \frac{WL^2}{6}$$

which with reduction gives

$$S^2 = \frac{L^2}{3}$$

i.e.

$$\boxed{S = \frac{L}{\sqrt{3}}}$$

This is the symmetrical spacing of the Airy points, *i.e.* approximately 0.577 of the length of the bar. This is only valid for a bar supported at two points with no additional reference flats or other masses attached to it. Even when a bar is supported at the Airy points, its central length will be different to the case where it is unsupported due to the extra curvature of the bar. Figure C.2 shows the difference dL in length between a bar which is unsupported and one which rests on supports positioned a distance a away from the end faces (L - S = $2a$). Note that supporting at the Airy positions (a = 0.211) causes a change in length of dL = -0.4 nm, which is negligible. The support positions corresponding to a = 0.185 for which there is the minimum change in length are termed the Bessel points.

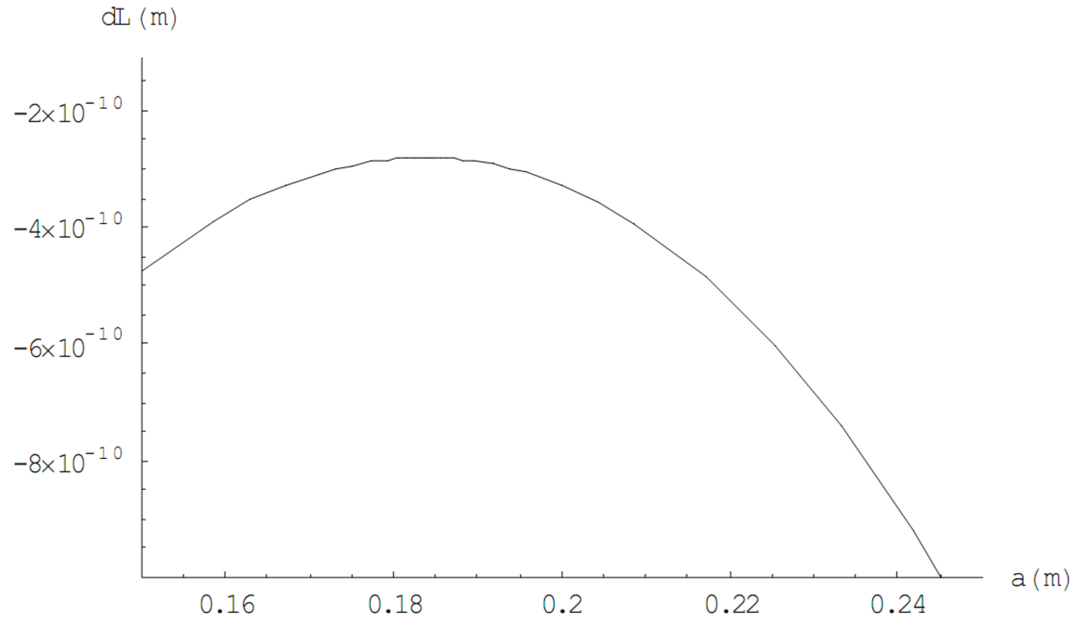

***Figure C.2** - Effect of support point position,* a, *on change in length, dL, of bar from unsupported state for a 1 m bar.*

C.3 COMPENSATION FOR MASS OF WRUNG FLAT

When a reference flat is wrung to one end face of a bar, this adds additional bending and will cause the bar supported at the Airy points to exhibit a parallelism error. Techniques for compensating for the extra mass of the flat include supplying an additional lifting force by means of weights or levers which effectively cancels out the weight of the flat [**2**] or by moving the support points towards the ends of the bar [**3**].

The latter solution has been adopted as being easier to implement and is detailed below.

Consider the bar and reference flat (platen) shown in figure C.3.

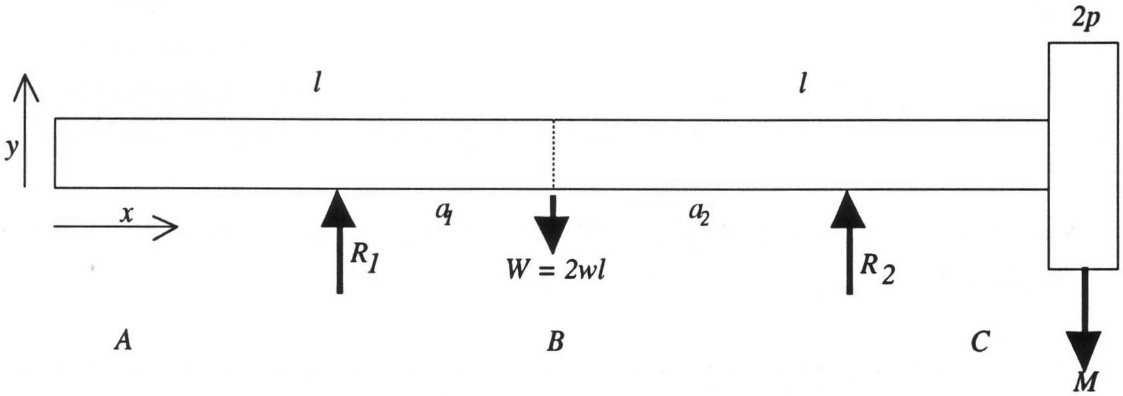

Figure C.3 *- Bar supported at new support points with flat attached to one face*

The supports are positioned at $x = l - a_1$ and $x = l + a_2$, with l being the half-length of the bar. As before, applying Bernoulli-Euler bending theory to the three regions gives three equations

$$EI\frac{d^2y}{dx^2}\bigg|_A = \frac{wx^2}{2} \qquad \text{for } 0 < x \le l - a_1 \qquad \text{(C.13)}$$

$$EI\frac{d^2y}{dx^2}\bigg|_B = \frac{wx^2}{2} - R_1(x - l + a_1) \qquad \text{for } l - a_1 < x \le l + a_2 \qquad \text{(C.14)}$$

$$EI\frac{d^2y}{dx^2}\bigg|_C = \frac{wx^2}{2} - R_1(x - l + a_1) - R_2(x - l - a_2) \qquad \text{for } l + a_2 < x \le 2l \qquad \text{(C.15)}$$

Integrating (C.13) (C.14) and (C.15) and determining arbitrary constants by continuity at support points, gives

$$EI\frac{dy}{dx}\bigg|_A = \frac{wx^3}{6}$$

$$EI\left.\frac{dy}{dx}\right|_B = \frac{wx^3}{6} - \frac{R_1}{2}(x - l + a_1)^2$$

$$EI\left.\frac{dy}{dx}\right|_C = \frac{wx^3}{6} - \frac{R_1}{2}(x - l + a_1)^2 - \frac{R_2}{2}(x - l - a_2)^2$$

This means that the angle between the end faces, α, is given simply by

$$\left.\frac{dy}{dx}\right|_{C,\,x=2l}$$

Thus

$$EI\alpha = \frac{4}{3}wl^3 - \frac{1}{2}(R_1 + R_2)(l^2 + a_1a_2) - \frac{1}{2}(R_1a_1 - R_2a_2)(2l + a_1 - a_2) \quad \text{(C.16)}$$

Now, resolving vertically, $R_1 + R_2 = W + M$ and taking moments about the centre of the bar gives $R_1a_1 - R_2a_2 = -M(l + p)$, substituting into (C.16) gives

$$2EI\alpha = W\left(\frac{l^2}{3} - a_1a_2\right) + M\left\{(l + p + a_1)(l + p - a_2) - p^2\right\} \quad \text{(C.17)}$$

To check the previous derivation for the Airy points, setting $M = 0$, $a_1 = a_2$ does indeed give the same solution for the positions of the supports.

To see the effect of supporting the bar and flat at the unmodified Airy points, the excess tilt of the ends of the bar can be calculated from

$$\alpha = \frac{Ml^2}{3EI}\left(1 + \frac{3p}{l}\right)$$

The flats are 70 mm diameter, 15 mm thick and have a density of 7800 kg m^{-3}. This gives values of M = 0.4503 kg, p = 7.5 x 10^{-3} m, I = 1.1923 x 10^{-8} m^4, and for steel, Youngs modulus, E = 203 GPa. For a 1 m bar, l = 1 m , this gives a value for α of 6.34 x 10^{-5} radians. Converting this to a change of length across the face of the bar gives a value of 1.4 µm, or over 4 fringes. To correct this, the two supports must be moved either symmetrically, or by moving just one support.

Let $$\frac{M}{W} = \frac{np}{l}$$

where n is the ratio of the cross-section of the flat to the cross-section of the bar, assuming that the bar and flat are made of the same material, as required to minimise the phase correction.

From (C.17), setting $\alpha = 0$, dividing by W and substituting $\frac{M}{W} = \frac{np}{l}$ gives

$$\left(\frac{l^2}{3} - a_1 a_2\right) + \frac{np}{l}\left\{(l + p + a_1)(l + p - a_2) - p^2\right\} = 0$$

There are 4 solutions for the positions of the support points: the first two being non-symmetrical and the remaining two being symmetrical and identical except for a change of sign. The non-symmetrical solutions leave one of the supports at its Airy point, and the solution of the above equation gives the position of the other support. For the symmetrical solution, both of the supports are moved outwards from their Airy points and retain their symmetrical placing about the centre of the bar.

Case (i), support 2 is unmoved, substituting $a_2 = \frac{l}{\sqrt{3}}$ in (C.17)

$$\left(\frac{l^2}{3} - \frac{a_1 l}{\sqrt{3}}\right) + \frac{np}{l}\left\{(l + p + a_1)\left(l + p - \frac{l}{\sqrt{3}}\right) - p^2\right\} = 0$$

Separating terms in a_1

$$\frac{l^2}{3} - a_1\frac{l}{\sqrt{3}} + \frac{np}{l}\left\{(l + p)\left(l + p - \frac{l}{\sqrt{3}}\right) - p^2\right\} + a_1\frac{np}{l}\left(l + p - \frac{l}{\sqrt{3}}\right) = 0$$

$$\frac{l^2}{3} + \frac{np}{l}\left\{(l + p)\left(l + p - \frac{l}{\sqrt{3}}\right) - p^2\right\} = a_1\left(\frac{l}{\sqrt{3}} - \frac{np}{l}\left(l + p - \frac{l}{\sqrt{3}}\right)\right)$$

$$a_1 = \frac{\frac{l^2}{3} + \frac{np}{l}\left\{(l + p)\left(l + p - \frac{l}{\sqrt{3}}\right) - p^2\right\}}{\frac{l}{\sqrt{3}} - \frac{np}{l}\left(l + p - \frac{l}{\sqrt{3}}\right)}$$

Removing a common factor of $\frac{l}{\sqrt{3}}$ gives

$$a_1 = \frac{l}{\sqrt{3}}\left\{\frac{\frac{l}{\sqrt{3}} + \frac{\sqrt{3}}{l}\frac{np}{l}\left\{(l + p)\left(l + p - \frac{l}{\sqrt{3}}\right) - p^2\right\}}{\frac{l}{\sqrt{3}} - \frac{np}{l}\left(l + p - \frac{l}{\sqrt{3}}\right)}\right\}$$

Dividing gives

$$a_1=\frac{l}{\sqrt{3}}\left\{1+\frac{\left[\frac{-\sqrt{3}}{l}(l+p)-1\right]\left[-\frac{np}{l}\left(l+p-\frac{l}{\sqrt{3}}\right)\right]-\frac{p^2\sqrt{3}np}{l^2}}{\frac{l}{\sqrt{3}}-\frac{np}{l}\left(l+p-\frac{l}{\sqrt{3}}\right)}\right\}$$

Multiplying top and bottom by $\sqrt{3}$, separating factors and rearranging gives

$$a_1=\frac{l}{\sqrt{3}}\left\{1+\frac{\frac{np}{l}\left(l+p-\frac{l}{\sqrt{3}}\right)\left(\sqrt{3}+3+\frac{3p}{l}\right)-\frac{3p^2}{l}}{l-\frac{np}{l}\left(\sqrt{3}l+\sqrt{3}p-l\right)}\right\}$$

Multiplying and collecting terms, dividing by l gives

$$a_1=\frac{l}{\sqrt{3}}\left\{\frac{\frac{np}{l}\left[2+\frac{6p}{l}\right]}{1-\frac{np}{l}\left(\sqrt{3}+\frac{\sqrt{3}p}{l}-1\right)}\right\}$$

$$a_1=\frac{l}{\sqrt{3}}\left\{\frac{\frac{2np}{l}\left[1+\frac{3p}{l}\right]}{1+\frac{np}{l}\left(1-\sqrt{3}-\frac{\sqrt{3}p}{l}\right)}\right\}$$

Thus with $$f(\chi)\equiv 1+\frac{2np/l(1+3p/l)}{1+np/l(1+\chi+\chi p/l)}$$

$$a_1=\frac{l}{\sqrt{3}}f\left(-\sqrt{3}\right)$$

For **case (ii)**, support 1 is unmoved, substituting $a_1=\frac{l}{\sqrt{3}}$ in (C.17) gives a similar solution to case (i), though because the signs of a_1 and a_2 are reversed, the sign of the radical is also reversed in the solution, *i.e.*

$$a_1=\frac{l}{\sqrt{3}}f\left(\sqrt{3}\right)$$

For **case (iii)**, both supports are moved symmetrically, substituting $a_1=a_2=a$ in (C.17) gives

$$\frac{l^2}{3}-a^2+\frac{np}{l}\left\{(l+p+a)(l+p-a)-p^2\right\}=0$$

Separating terms in a

$$\frac{l^2}{3}-a^2+\frac{np}{l}\left\{(l+p)(l+p)-a^2\right\}=0$$

$$a=\frac{l}{\sqrt{3}}\sqrt{\frac{1+\frac{3np}{l^3}\left(l^2+2pl\right)}{1+\frac{np}{l}}}$$

Dividing

$$a=\frac{l}{\sqrt{3}}\sqrt{1+\frac{\left[3\left(1+\frac{2p}{l}\right)-1\right]\frac{np}{l}}{1+\frac{np}{l}}}$$

$$a=\frac{l}{\sqrt{3}}\sqrt{1+\frac{\frac{2np}{l}\left(1+\frac{3p}{l}\right)}{1+\frac{np}{l}}}$$

$$a=\frac{l}{\sqrt{3}}\sqrt{f(0)}$$

Strictly, $a=\pm\frac{l}{\sqrt{3}}\sqrt{f(0)}$ though these two solutions correspond to the two choices of labelling the supports, *i.e.* they are the same physical solution.

In summary, setting α to zero in (C.17) allows for three solutions:

(i) Support 2 remains at the Airy position, and support 1 moves to a new position

$$a_2=\frac{l}{\sqrt{3}} \quad , \quad a_1=\frac{l}{\sqrt{3}}f\left(-\sqrt{3}\right)$$

(ii) Support 1 remains at the Airy position and support 2 moves to a new position

$$a_1=\frac{l}{\sqrt{3}} \quad , \quad a_2=\frac{l}{\sqrt{3}}f\left(\sqrt{3}\right)$$

(iii) Both supports move by equal amounts to new symmetrical positions

$$a_1 = a_2 = a = \frac{l}{\sqrt{3}}\sqrt{f(0)}$$

where $$f(\chi) \equiv 1 + \frac{2np/l(1+3p/l)}{1+np/l(1+\chi+\chi p/l)}$$

Suitable tolerances on the positioning of the supports may be calculated by differentiating (C.17) with respect to a, this will be performed for the symmetrical solution (case (iii)).

Substituting $a_1 = a_2 = a$ in (C.17) gives

$$2EI\alpha = W\left(\frac{l^2}{3} - a^2\right) + M\{(l+p+a)(l+p-a) - p^2\}$$

$$2EI\alpha = \frac{Wl^2}{3} - Wa^2 + M\left(l^2 + 2lp - a^2\right)$$

Differentiating with respect to a gives

$$2EI\delta\alpha = -2a(W+M)\delta a$$

Hence $$\delta a = \frac{EI}{a(W+M)}\delta\alpha \qquad (18)$$

For a 1 m bar, for a maximum value of $\delta\alpha$ of 1.126 x 10^{-6} which corresponds to the value of 1 μin (0.025 μm) error chosen by Williams, δa = 2.4 x 10^{-3}, or 2.4 mm. This is better than the tolerance for the general case for which Williams calculated a value of 0.7 mm. Thus the use of symmetrical support positions is preferable, for which positioning within 2.4 mm is required.

Thus by accurate positioning of the support positions, the additional bending may be altered in such a way that the end faces of the bar remain vertical and parallel. The effect of this additional bending on the length of the bar will now be examined.

C.4 EFFECT OF FLEXURE OF BARS ON THEIR LENGTH

The effect on the measured length of the bar is measured on the neutral axis of the bar which runs through the centre of the bar. For a section of the bar, length dx, with gradient θ the change in length compared to the free state is given by $\frac{\theta^2}{2}dx$, and $\theta = \frac{dy}{dx}$. Thus the total change in length along the whole bar is given by

$$\int_0^{2l} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

It is possible to perform this integral, substituting for $\frac{dy}{dx}$ from equations derived earlier, but a simple order of magnitude estimate shows that this is not required as the overall change in length is negligible. Since

$$\int_0^{2l} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \le l\left(\left.\frac{dy}{dx}\right|_{\max}\right)^2$$

a maximum value for the change in length due to bending may be calculated. Figure C.4 shows the variation in the vertical position of the neutral axis of a 1m bar with a flat wrung on, supported at the modified symmetrical Airy points and the slope of the bar. The maximum slope is seen to be 8 x 10^{-6} at approximately 0.7 m from the free end of the bar. Thus the maximum change in length of the bar is 6.4 x 10^{-11} m (0.002 fringe), *i.e.* negligible.

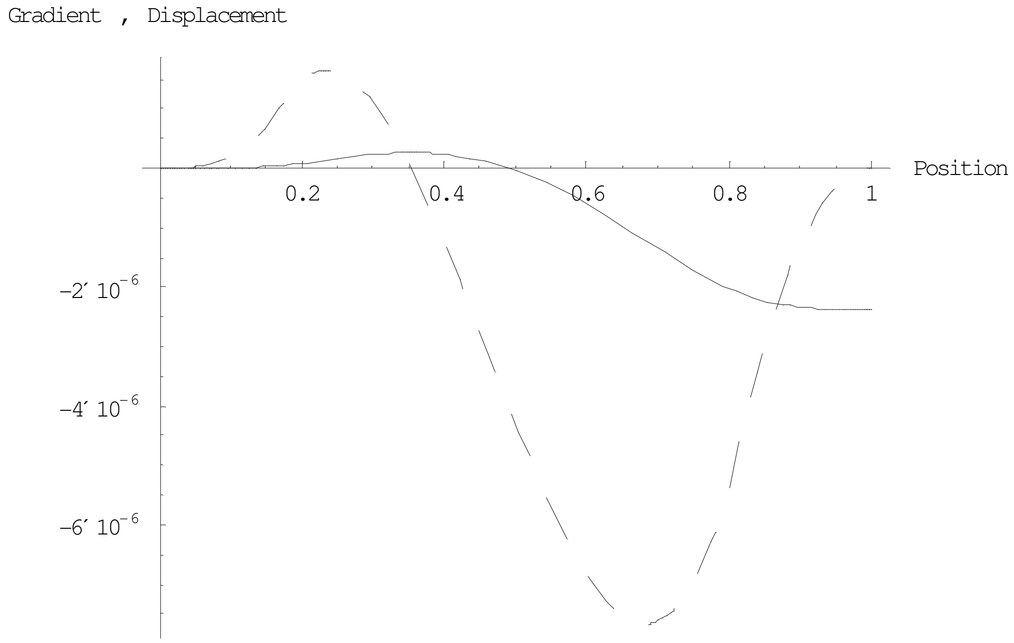

Figure C.4 *- Variation in vertical position and gradient (dashed line) of the neutral plane of a 1 m bar, supported at modified Airy points*

REFERENCES FOR APPENDIX C

[**1**] Rolt F H *Gauges and Fine Measurement* (MacMillan & Co.: London) (1929) 15-16

[**2**] Bayer-Helms F Beigung von Endmaßen bei horizontaler Lagerung auf Scheiden *PTB Mitteilungen* (1967) **1/67** 25-30 **2/67** 124-130],

[**3**] Williams D C The parallelism of a length bar with an end load *J. Sci. Instrum.* (1962) **39** 608-610