APPENDIX D

PRISMATIC COMPRESSION OF LENGTH BARS

(These results were derived by mathematicians in NPL's Division of Information Technology and Computing).

With reference to Timoshenko & Goodier [1], the forces in the body are

$$X = Y = 0, \quad Z = -\rho g$$

The differential equations of equilibrium ((127) of Timoshenko & Goodier) are satisfied by

$$\sigma_z = \rho g(z - l), \qquad \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$$

i.e. by assuming that on each cross section we have a uniform compression produced by the upper portion of the bar (see figure D.1).

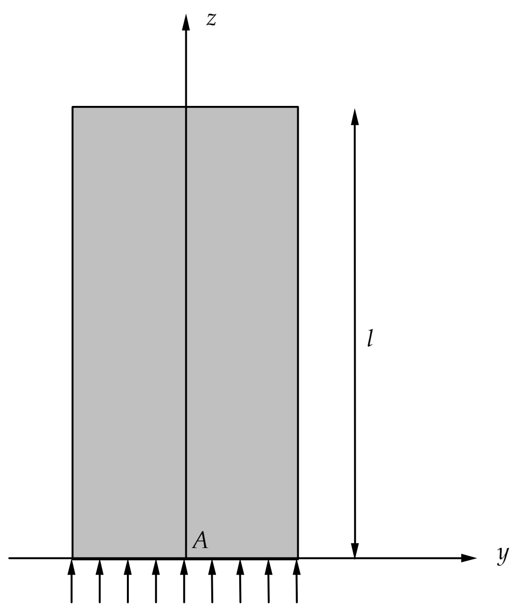

Figure D.1 *- Compression on cross section of bar*

Hooke's law gives

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{\sigma_z}{E} = \frac{\rho g}{E}(z - l) \tag{D.1}$$

$$\varepsilon_x = \varepsilon_y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\upsilon\frac{\rho g}{E}(z-l) \tag{D.2}$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{D.3}$$

Integrating (D.1) gives

$$w = \frac{\rho g z^2}{2E} - \frac{\rho g l z}{E} + w_0 \tag{D.4}$$

where w_0 does not depend on z, *i.e.* $w_0 = w_0(x,y)$. Substituting (D.4) into the second and third of equations (D.3) we find

$$\frac{\partial w_0}{\partial x} + \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial w_0}{\partial y} + \frac{\partial v}{\partial z} = 0$$

from which

$$u = -z\frac{\partial w_0}{\partial x} + u_0, \qquad v = -z\frac{\partial w_0}{\partial y} + v_0 \tag{D.5}$$

where u_0 and v_0 are functions of x and y only. Substituting these expressions into (D.2) we find

$$-z\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\frac{\upsilon\rho g}{E}(z-l), \qquad -z\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\frac{\upsilon\rho g}{E}(z-l)$$

Equating polynomial terms in z, we have

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2 w_0}{\partial y^2} = \frac{\upsilon\rho g}{E}, \qquad \frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} = \frac{\upsilon\rho g l}{E} \tag{D.6}$$

Substituting expressions (D.5) into the first of equations (D.3), we find

$$-2z\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0$$

and since u_0 and v_0 do not depend on z, we have

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0, \qquad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \tag{D.7}$$

From the six equations in (D.6) and (D.7) we can write general expressions for u_0, v_0 and w_0. All these equations are satisfied by

$$u_0 = \frac{\upsilon\rho g l x}{E} + \delta y + \delta_1$$

$$v_0 = \frac{\upsilon\rho g l y}{E} - \delta x + \gamma_1$$

$$w_0 = \frac{\upsilon\rho g}{2E}\left(x^2 + y^2\right) + \alpha x + \beta y + \gamma$$

Now substituting these expressions into equations (D.4) and (D.5), the general expressions for the displacements are

$$u = -\frac{\upsilon\rho g x z}{E} - \alpha z + \frac{\upsilon\rho g l x}{E} + \delta y + \delta_1$$

$$v = -\frac{\upsilon\rho g y z}{E} - \beta z + \frac{\upsilon\rho g l y}{E} - \delta x + \gamma_1$$

$$w = \frac{\rho g z^2}{2E} - \frac{\rho g l z}{E} + \frac{\upsilon\rho g}{2E}\left(x^2 + y^2\right) + \alpha x + \beta y + \gamma$$

The six arbitrary constants are determined from the conditions at the support. We prevent translatory movement of the bar by fixing the centroid A of the lower end of the bar so that for $x = y = z = 0$, we have $u = v = w = 0$. We eliminate rotation of the bar about axes through A parallel to the x and y axes by fixing an element of the z axis at A. Then $\partial u / \partial z = \partial v / \partial z = 0$ at A. We avoid the possibility of rotation about the z axis by ensuring that $\partial v / \partial x = 0$ at the point A. From these six conditions at A, we find that all the arbitrary constants (α, β, γ, γ_1 and δ_1) vanish. Thus we are left with

$$u = \frac{\upsilon\rho g x}{E}(l - z)$$

$$v = \frac{\upsilon\rho g y}{E}(l - z)$$

$$w = \frac{\rho g z^2}{2E} + \frac{\upsilon\rho g}{2E}\left(x^2 + y^2\right) - \frac{\rho g l z}{E}$$

Hence for the point $(x,y,z) = (0,0,l)$ initially, we have, after deformation:

$$z = l + w(0,0,l) = l - \frac{\rho g l^2}{2E}$$

Hence the change in length of a bar when standing vertically, due to its own weight, is given by

$$-\frac{\rho g l^2}{2E}$$

Using this equation and average values for steel of ρ = 7800 kg m^{-3}, E = 208 GPa, g= 9.8 m s^{-1} produces the graph shown in figure D.2.

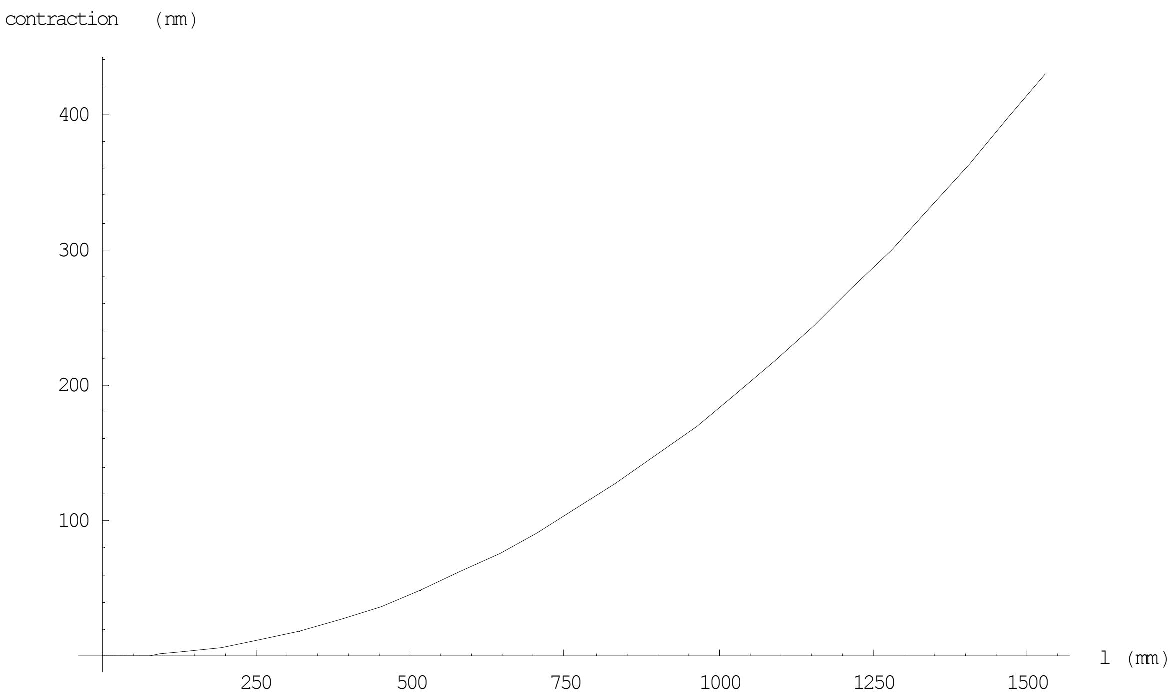

Figure D.2 *- Contraction of a steel length bar, standing vertically*

REFERENCES FOR APPENDIX D

[**1**] Timoshenko S & Goodier J N *Theory of Elasticity* (McGraw-Hill, New York) (1951) 246