CHAPTER 10

UNCERTAINTY OF MEASUREMENTS

"It is much easier to recognise error than to find truth; error is superficial and may be corrected; truth lies hidden in the depths"
Goethe

10.1 THE NATURE OF ERRORS

When a length bar is measured in the interferometer, the result of the length calculation will be subject to an uncertainty due to the design and operation of the instrument. The total uncertainty will be the sum of many contributing uncertainties. These may be due to uncertainties in measured physical quantities, imperfections in the theory describing the interferometer operation, or departure from the theory in the real world. It is important when using the interferometer to measure a bar, to be aware of the uncertainty in the measurement.

10.1.1 The 'orthodox' theory of errors

According to orthodox views of error theory [**1**], there are 2 basic types of error: random and systematic. Random errors can be seen when the measured value of a physical quantity is different under nominally identical circumstances. Systematic errors can arise when a derived correction is applied to measured data, *e.g.* the refractive index correction.

The two types of error are very different in their effects on the measurement of length in the interferometer. If one makes sufficient measurements, the random uncertainties will be symmetrically distributed about a mean value, which, in the absence of systematic errors, will be the correct value. However, even when many measurements are made with systematic errors present, the calculated mean may be biased away from the true mean, especially if many of the systematic errors add with the same sign, and hence do not cancel each other.

There will also be unknown sources of error, whose nature is unknown. These may cause the cautious experimenter to overestimate the effects of one or other of the types of uncertainty when trying to make allowance for these errors.

Another distinction in the sources of error can be made for a length measuring interferometer. There will be some sources of error which are inherent in the basic design of the instrument, which will contribute an error, even if a 'zero-length' object were measured. Other sources of error will depend on the length of the object being measured, *i.e.* they are length dependent. It is useful to quote the total uncertainty of the instrument in a form which separates these two types of error:

$$U = a + bL \tag{10.1}$$

where U is the total uncertainty, a is the inherent uncertainty (random and systematic), b is the length dependent uncertainty (random and systematic), and L is the length being measured. In order to be able to compare random and systematic errors in this way, a common form of reference must be established.

As measurements made by the interferometer will be used at the top of the UK's hierarchy of traceable length measurements, the calculation and expression of the uncertainty of the result must be made with reference to standard statistical treatments of uncertainty. The basis of the following error analysis is NAMAS document NIS 3003 [**2**]. This is similar to the draft WECC document 19-1990 [**3**].

10.1.2 Combination of errors

In the orthodox view, uncertainties or errors are usually combined in quadrature [**2**,**4**]

$$U_{TOT} = \sqrt{\sum_i U_i^2} \tag{10.2}$$

This is only correct if the estimates of the errors, U_i, are equally weighted, *i.e.* they have the same confidence intervals. For random errors which are normally distributed, this method is correct, as the representative uncertainty of a set of observations is the variance, σ, which always corresponds to a confidence interval of 0.68, or 68% for a normal (Gaussian) distribution. However, the confidence interval of a distribution or errors of a systematic nature is not always the same.

For high accuracy calibrations, such as those offered by the interferometer, it is usual to take a confidence interval of 0.95 (95%) to standardise the uncertainty of measurement when comparing measurements made using different instruments.

10.1.3 Random errors

For the purposes of this error analysis it is assumed that the random uncertainties in a set of *N* observations or measurements are from a larger distribution, which is itself assumed to be Gaussian. In the absence of sufficient data, the standard deviation can be estimated from the range, *R*, of the measured values by

$$\sigma = \pm \kappa R \tag{10.3}$$

where κ is approximated by

$$\kappa \approx \frac{1}{\sqrt{N}} \tag{10.4}$$

The standard error of the mean of the *N* observations is given by

$$SEOM = \pm \frac{\sigma}{\sqrt{N}} \tag{10.5}$$

To convert this to a confidence interval, the *SEOM* is multiplied by a factor *t,* the *student t factor*, which depends on the required confidence interval and the number of measurements made. Values of *t* are tabulated in the literature [**5**,**2**,**6**]. When the behaviour of an instrument or uncertainty is well known, either by having made a large number of measurements, or by assuming an uncertainty from the specifications of the instrument, it is then correct to take a value of *t* corresponding to an infinite number of measurements. At a confidence interval of 95%, this value is *t* = 1.96 (sometimes referred to as *k*).

Thus the confidence interval for random uncertainties is given by

$$C_R = \pm \frac{t\sigma}{\sqrt{N}} \tag{10.6}$$

and the total random uncertainty is given by

$$U_R = \sqrt{\sum\nolimits_R C_R^{\,2}} \tag{10.7}$$

10.1.4 Systematic errors

When assessing the effect of systematic errors, an estimate of the standard deviation of a systematic effect on the mean value of the quantity being measured should be used. If this is not possible, then realistic limits for the systematic contribution should be estimated. When a number of error distributions are combined, the Central Limit Theorem states that the overall combined distribution will tend towards a Gaussian. The accuracy of the approximation will depend on the form of the individual distributions and their standard deviations. If it is assumed that a systematic error lies within the bounds *-R/2* to *+R/2*, then an approximate standard deviation for this distribution will be

$$\sigma = \frac{R}{2\sqrt{3}} \tag{10.8}$$

To convert this to a confidence interval, it is multiplied by a factor k_S, which is dependent on the required confidence level. For a 95% confidence level, k_S = 1.96. Thus

$$C_S = \pm \frac{k_s R}{2\sqrt{3}} \tag{10.9}$$

and the overall systematic uncertainty is given by

$$U_S = \sqrt{\sum_S C_S^2} \tag{10.10}$$

According to the NAMAS guidelines, provided that $k_S > 1.8$, the probability of the error falling within $\pm\, C_S$ will always be greater than for a truly Gaussian distribution of the same standard deviation.

10.2 BIPM RECOMMENDATIONS ON ERROR ASSESSMENT

Many scientific and industrial activities require only rough-and-ready ‘uncertainty’ estimates using simple techniques. However metrologists and others making fundamental physical measurements require a rigorous and objective (*i.e.* demonstrably realistic) theory of errors on which to base accurate estimates of uncertainty. The BIPM has issued recommendations for the estimation of experimental uncertainty [**7**].

A summary of their recommendations follows.

1. The uncertainty in the result of a measurement generally consists of several components which may be grouped into two categories according to the way in which their numerical value is estimated:

A - those which are evaluated by statistical methods,
B - those which are evaluated by other means.

There is not always a simple correspondence between the classification into categories A or B and the previously used classification into "random" and "systematic" uncertainties. The term "systematic uncertainty" can be misleading and should be avoided.

Any detailed report of the uncertainty should consist of a complete list of the components, specifying for each the method used to obtain its numerical value.

2. The components in category A are characterised by the estimates s_i^2, (or the estimated "standard deviations" s_i) and the number of degrees of freedom v_i. Where appropriate, the estimated covariances should be given.

3. The components in category B should be characterised by quantities u_j^2, which may be considered as approximations of the corresponding variances, the existence of which is assumed. The quantities u_j^2 may be treated like variances and the quantities u_j like standard deviations. Where appropriate, the covariances should be treated in a similar way.

4. The combined uncertainty should be characterised by the numerical value obtained by applying the usual method for combination of variances. The combined uncertainty and its components should be expressed in the form of "standard deviations".

5. If, for particular applications, it is necessary to multiply the combined uncertainty by a factor to obtain an overall uncertainty, the multiplying factor must always be stated.

10.3 COMPARISON OF 3 THEORIES OF ERROR AND RECOMMENDATIONS

Colclough [**2**] compared the orthodox and BIPM recommendations on errors and considered a third theory, the "Randomatic Theory of Errors" in which all errors are treated in the same way as random errors in the orthodox theory. In his analysis, he stated that all errors could be divided into 4 classes, with each error belonging to one class and one class only.

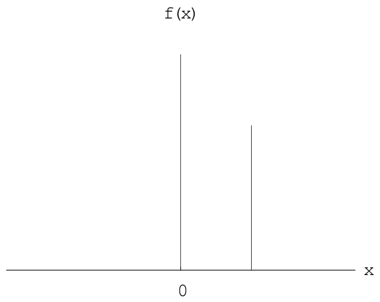
The four classes (illustrated in figure 10.1) illustrate the way in which the observed results of an experiment behave when the experiment is repeated several times:

Class 1 - each result may differ from the true value by the same amount and with the same sign, *i.e.* the error is constant,

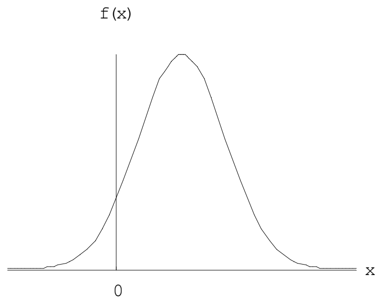
Class 2 - each error may vary randomly realising a stable distribution with a non-zero mean,

Class 3 - each error may vary randomly realising a stable distribution with a zero mean,

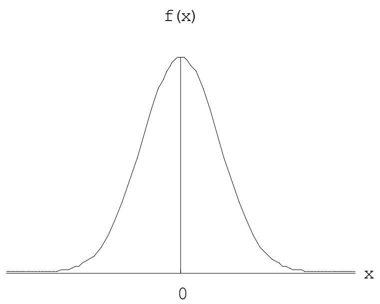
Class 4 - each error may vary non-randomly (*e.g.* cyclically or by failing to produce convergent distributions, sometimes referred to as a 'locally systematic error')



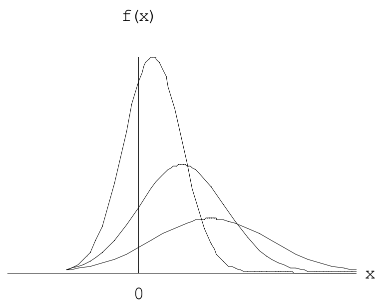
Class 1 error



Class 2 error



Class 3 error



Class 4 error

Figure 10.1 *- Four classes of experimental error*

Colclough showed that all three theories of errors were flawed: the orthodox theory is not rigorous enough in the combination of errors and there is uncertainty as to which results contain random errors; the BIPM technique uses approximations of variances and is still controversial; the Randomatic theory uses unrealistic distributions and raises controversial questions in terms of the law of error propagation. The subject of error theory still raises controversy particularly since the experimenter has to assess probabilities in the absence of both statistical data and real data.

It is thus difficult to choose a particular technique for calculating the uncertainty budget for the new interferometer. However all the above theories make recommendations which are of use in this situation.

Recommended analysis

Firstly, the whole of the experimental procedure should be defined, and all sources of error identified. A confidence level is chosen, beyond which errors will be regarded as improbable. This confidence level must be clearly stated. Each error is then attributed to a class: random/systematic or class 1 to class 4. This decision is often taken in the absence of trial data by careful consideration of the conditions.

In the case of class 4 errors, either they should be reduced by modification of the experimental technique, or maximum errors of the quantity concerned are computed - these should be treated as systematic errors.

Next, the maximum and minimum possible or likely values of the class 1 errors and the constant components of class 2 errors are estimated, either by reference to assumed specifications or by examining error distributions. These errors are propagated through to the final measurement uncertainty. These components are added arithmetically to give an overall systematic uncertainty in the final result.

All the class 2 and class 3 sources of random errors are identified and propagated through to the final measurement uncertainty. These components are combined in quadrature to obtain a standard deviation for the random error component.

The systematic uncertainties are then used to define upper and lower limits for the mean of the overall random uncertainty giving two worst-case distributions. The upper and lower confidence limits of these two distributions are used to arrive at a final estimate of the uncertainty.

10.4 SOURCES OF UNCERTAINTY

The individual sources of error which affect an individual length measurement made by the interferometer will now be examined. These include errors in the measurement of physical variables in which there may be several contributing uncertainties and also errors due to the design and operation of the interferometer. For each source of error, its magnitude will be estimated along with its effect (random or systematic) including whether or not it is length dependent. The uncertainties are quoted as uncertainties in physical units followed by the corresponding uncertainties converted to length units, where L is the length of the bar, in metres. The class of error is also identified for both the random/systematic and class 1...class 4 schemes, labelled as *e.g*. R 3 for a pseudo-random class 3 uncertainty.

Where error sources relate to manufacturer-specified accuracies or for calibrations of equipment, these are for a confidence level not less than 95%. Thus the effect of these error may be overestimated by a factor of 1.96 in the final calculation - this is tolerable, since in many cases these errors are small and an over-estimation of the final error is better than under-estimation.

10.4.1 Air pressure measurement

The pressure is measured by a Druck DPI140 pressure transducer (see § 7.3.1). The instrument is calibrated at yearly intervals against NPL primary standards. The measurement is performed with dry air over 3 pressure cycles. The deviation of the measured pressure from the accurately known supplied pressure is noted at 9 points during both rising and falling pressure conditions. The calibration is performed at approximately 20 °C.

The DPI140 measures the pressure inside the chamber via a sample pipe. The pipe is at approximately the same height as the length bar being measured. The optical beam diameter at the length bar is 80 mm. The interferometer chamber contains moist air from the room at relative humidity (RH) 50% (± 5%).

The following sources of uncertainty have been identified:

Accuracy of NPL working Standard	R 3	± 0.05 mbar	± 1.34 x 10^{-8} *L*
Maximum departure of DPI readings from mean during up/down cycle	R 3	± 0.06 mbar	± 1.61 x 10^{-8} *L*
Error in reading at 50 % RH (± 5%), 20 °C	S 1	+ 0.057 mbar	+ 1.53 x 10^{-8} *L*
due to water vapour	R 3	± 0.0057 mbar	± 1.53 x 10^{-9} *L*
Pressure gradient due to gravity, across beam diameter	S 1	+ 0.0034 mbar	± 9 x 10^{-10} *L*
Resolution of DPI140 instrument	R 3	± 0.01 mbar	± 2.68 x 10^{-9} *L*
TOTAL	R 3	± 0.0789 mbar	± 2.12 x 10^{-8} *L*
TOTAL	S 1	+ 0.0604 mbar	+ 1.62 x 10^{-8} *L*

10.4.2 Air temperature measurement

The temperature of the air in the chamber is measured using a PRT. The PRT is placed in a heatsink and is positioned near to the measurement beam, usually behind the bar being measured. The temperature is measured by measuring the resistance of the PRT using a resistance bridge. The PRT is calibrated at 2 yearly intervals by Temperature Section, NPL, against the water triple point and gallium melting point. Equations conforming to the ITS-90 specification [**8**] are used to interpolate between these two standard temperatures. The bridge is calibrated monthly by using it to measure the resistance of a standard 100 Ω resistor, which is itself calibrated yearly. The PRTs are checked every 6 months by using them to measure the temperature of a water triple point cell.

The following sources of uncertainty have been identified:

Resolution of resistance bridge	R 3	± 10 μΩ = ± 0.03 mK	± 2.78 x 10^{-11} L
Resistance bridge accuracy:	R 3	± 1 ppm ± 10 μΩ = ± 101 μΩ = ± 0.3 mK	± 2.78 x 10^{-10} L
Accuracy of external standard resistor	R 3	± 8 μΩ = ± 0.024 mK	± 2.22 x 10^{-11} L
<u>PRT calibration</u>			
Water triple point accuracy	R 3	± 0.5 mK	± 4.65 x 10^{-10} L
Gallium melting point accuracy	R 3	± 0.5 mK	± 4.65 x 10^{-10} L
Interpolating equations	R 3	± 0.13 mK	± 1.21 x 10^{-10} L
Drift between calibrations	R 3	< ± 0.5 mK	± 9.3 x 10^{-10} L
TOTAL	R 3	± 0.926 mK	± 8.58 x 10^{-10} L

10.4.3 Air humidity measurement

The humidity of the air inside the chamber is measured by extracting a sample of the air through a Michell S3000 dewpoint hygrometer. The S3000 is calibrated by a NAMAS accredited laboratory against standard humidity gases at a flow rate of 0.5 l min^{-1}. The voltage output of the S3000 is read by an IEEE voltmeter. The voltmeter is calibrated at the 0 V and 999.9 mV points using a standard voltage generator. The agreement at interpolated voltages is within ± 0.2 mV. Magnus' equation [**9**] is used to convert dewpoint into partial pressure. This has been compared with other techniques, such as Goff-Gratch [**10**] and found to be in agreement to within 2% RMS over the range 0 to 30 °C.

The following sources of uncertainty have been identified:

Accuracy of dewpoint of standard humidity gases	R 3	± 0.25 °C DP = ± 0.207 mbar	± 5.65 x 10^{-9} *L*
Resolution of S3000	R 3	± 0.1 °C DP = ± 0.083 mbar	± 2.26 x 10^{-9} *L*
Resolution of IEEE voltmeter	R 3	± 0.1 mV = ± 0.01 °C DP	± 2.26 x 10^{-10} *L*
Accuracy of IEEE voltmeter calibration	R 3	± 0.2 mV = ± 0.02 °C DP	± 4.52 10^{-10} *L*
Accuracy of standard voltage source	R 3	± 0.2 mV = ± 0.02 °C DP	± 4.52 10^{-10} *L*
Humidity gradient between sample point and beam	R 3	± < 0.05 °C DP	± 1.13 x 10^{-9} *L*
Accuracy of Magnus' eqn	R 3	± 0.2 °C DP	± 4.52 x 10^{-9} *L*
TOTAL	R 3	± 0.340 °C DP	± 7.70 x 10^{-9} *L*

10.4.4 Air CO_2 measurement & Edlén's equations

The CO_2 content of the air inside the chamber is measured by extracting a sample of the air (the same as used for the humidity measurement) through an Edinburgh Instruments GASCARD CO_2 meter. The GASCARD meter is calibrated at two points against standard gases with CO_2 concentrations of 0 ppm and 370 ppm CO_2 by volume. This calibration is performed yearly.

The following sources of uncertainty have been identified:

Resolution of GASCARD meter	R 3	± 18 ppm	$\pm 2.65 \times 10^{-9}\,L$
Accuracy of 0 ppm standard gas	R 3	± 1 ppm	$\pm 1.47 \times 10^{-10}\,L$
Accuracy of 370 ppm standard gas	R 3	± 30 ppm	$\pm 4.41 \times 10^{-9}\,L$
Interpolation between calibration points	R 3	± 5 ppm	$\pm 7.35 \times 10^{-10}\,L$
Variation in concentration between sample point and measurement beam	S 1	- 10 ppm	$- 1.47 \times 10^{-9}\,L$
TOTAL	R 3	± 35.4 ppm	$\pm 5.20 \times 10^{-9}\,L$
	S 1	-10 ppm	$-1.47 \times 10^{-9}\,L$
Accuracy of modified Edlén equation with CO_2	R 3	$\pm 1 \times 10^{-8}$	$\pm 1 \times 10^{-8}\,L$

10.4.5 Laser wavelength

The lasers are all frequency-stabilised helium-neon continuous wave lasers operating at 632.990876 nm (red), 543.516364 nm (green) and 611.970617 nm (orange). They are calibrated by direct comparison with NPL Primary lasers, one of which (at approximately 633 nm) represents the UK's realisation of the metre. The calibration is a beat frequency comparison so there is no correction for the refractive index of the air. The measured length of the bar is the length measured by the red wavelength as this has a lower overall uncertainty than the mean of the lengths measured by three wavelengths with equal weighting. The green and orange laser wavelength uncertainties are given here for comparison. The lasers are calibrated by direct frequency comparison against primary reference lasers at NPL. The primary lasers are stabilised by saturated absorption in molecular iodine at the following transitions:

632.991 398 22 nm	($\pm$ 2.5 x 10^{-11})	11-5 R(127) a_{13}
611.970 770 0 nm	($\pm$ 3 x 10^{-10})	9-2 R(47) a_7
543.516 333 1 nm	($\pm$ 2.5 x 10^{-10})	26-0 R(127) a_9

The uncertainties quoted for the wavelengths are the "estimated relative standard uncertainties", which are similar to 1 σ values.

The following sources of uncertainty have been identified:

RED			
Uncertainty of primary standard frequency	R 3	$\pm$ 2.5 x 10^{-11}	$\pm$ 2.5 x 10^{-11} *L*
Accuracy of calibration	R 3	$\pm$ 1 x 10^{-9}	$\pm$ 1 x 10^{-9} *L*
Variability (short-term) in stabilised test laser	R 3	$\pm$ 1.6 x 10^{-9}	$\pm$ 1.6 x 10^{-9} *L*
GREEN			
Uncertainty of primary standard frequency	R 3	$\pm$ 2.5 x 10^{-10}	$\pm$ 2.5 x 10^{-10} *L*
Accuracy of calibration	R 3	$\pm$ 1 x 10^{-9}	$\pm$ 1 x 10^{-9} *L*
Variability (short-term) in stabilised test laser	R 3	$\pm$ 9 x 10^{-9}	$\pm$ 9 x 10^{-9} *L*
ORANGE			
Uncertainty of primary standard frequency	R 3	$\pm$ 3 x 10^{-10}	$\pm$ 3 x 10^{-10} *L*
Accuracy of calibration	R 3	$\pm$ 1 x 10^{-9}	$\pm$ 1 x 10^{-9} *L*
Variability (short-term) in stabilised test laser	R 3	$\pm$ 3.3 x 10^{-9}	$\pm$ 3.3 x 10^{-9} *L*
RED WAVELENGTH TOTAL	R 3	$\pm$ 1.89 x 10^{-9}	$\pm$ 1.89 x 10^{-9} *L*

10.4.6 Mechanical - optical effects

No correction is made for the thickness of the wringing film since it is included in the definition of the length of the bar when measured interferometrically. However its variability can lead to a measurement uncertainty.

The following sources of uncertainty have been identified:

Effect of the source size (see § 4.1.3)	S 1	+ 4 µm diameter	$- 4 \times 10^{-13} L$
Source off axis (see § 4.1.3)	S 2	+ 50 µm	$- 5.6 \times 10^{-10} L$
	R 2	± 50 µm	$\pm 5.6 \times 10^{-10} L$
Chromatic aberration - focal length error	S 1	+ 0.47 mm	$- 4.4 \times 10^{-11} L$
Laser beam diffraction	S 1	+ 80 mm diameter	$- 2 \times 10^{-11} L$
Spherical aberration in collimation	S 1	$- 1 \times 10^{-9} L$	$- 1 \times 10^{-9} L$
Spherical aberration in de-collimator	S 1	$- 1 \times 10^{-9} L$	$- 1 \times 10^{-9} L$
Prismatic tilt at beamsplitter	S 1	+ 4.5 fringes	$- 5.1 \times 10^{-10} L$
Bar - beam alignment	R 3	± 2 fringes tilt	$\pm 1.62 \times 10^{-9} L$
Shortening due to support points	S 1	bar slope $< 8 \times 10^{-6}$	$- 6.4 \times 10^{-11} L$
Reference beam alignment	R 3	± 60 µm off axis	$\pm 8.0 \times 10^{-10} L$
Phase difference, dispersion and	S 2	-14 nm	-14 nm
surface roughness difference	R 2	± 27 nm	± 27 nm
Wringing film thickness	R 3	± 5 nm	± 5 nm
Accuracy of fringe fraction result and data analysis	R 3	± 0.016 fringe	± 5 nm
	R 3	TOTAL	± 28 nm $\pm 1.89 \times 10^{-9} L$
	S 1	TOTAL	-14 nm $- 3.25 \times 10^{-9} L$

10.4.7 Bar expansivity at 20 °C

Because it is not possible to make all measurements at exactly 20 °C, the measured length of the bar is corrected to 20 °C. This requires both a measurement of the temperature of the bar and also an estimate of its coefficient of thermal expansion. The temperature is measured using two PRTs which are in small copper blocks in thermal contact with the bar. The temperature of these PRTs is measured using a resistance bridge. The bridge is calibrated monthly by using it to measure an external 100 Ω standard resistor. The PRTs are calibrated at 2-yearly intervals and are checked every 6 months against a water triple point cell. The nominal coefficient of thermal expansion used for length bars (and also for gauge blocks over 100 mm in length) is α = 10.7 ppm K^{-1}. Variation in the value of α from bar to bar is estimated to be within ± 0.5 ppm K^{-1}. The temperature of the bar inside the chamber is 20 °C ± 0.03 °C.

The following sources of uncertainty have been identified:

Resolution of resistance bridge	R 3	± 10 μΩ = ± 0.03 mK	± 3.21 x 10^{-10} *L*
Resistance bridge accuracy	R 3	1 ppm ± 10 μΩ = ± 101 μΩ = ± 0.3 mK	± 3.21 x 10^{-9} *L*
Accuracy of standard resistor	R 3	± 8 μΩ = ± 0.024 mK	± 2.57 x 10^{-10} *L*
PRT calibration			
Water triple point accuracy	R 3	± 0.5 mK	± 5.35 x 10^{-9} *L*
Gallium melting point accuracy	R 3	± 0.5 mK	± 5.35 x 10^{-9} *L*
Interpolating equations	R 3	± 0.13 mK	± 1.39 x 10^{-9} *L*
Drift between calibrations	R 3	< ± 0.5 mK	± 5.35 x 10^{-9} *L*
Contact of PRT with bar	R 3	± 0.5 mK	± 5.35 x 10^{-9} *L*
Non-linear gradient at 20 °C	R 3	± 0.1 mK	± 1.07 x 10^{-9} *L*
TOTAL	R 3	± 1.05 mK	± 1.13 x 10^{-8} *L*
Accuracy of nominal α	R 3	± 0.5 ppm K^{-1} (@ 20.03 °C)	± 1.5 x 10^{-8} *L*

10.5 SUMMATION OF UNCERTAINTIES

In accordance with the guidelines, the random and systematic (class 3 and class 1) uncertainties are summed individually. The length dependent and length independent contributions are also treated separately. There are thus four separate error contributions:

S systematic, length independent
S_L systematic, length dependent
R random, length independent
R_L random, length dependent

The contributions to S and S_L are summed arithmetically, whereas the contributions to R and R_L are summed in quadrature. The random (class 3) uncertainties are then multiplied by a factor of 1.96 to obtain results at a confidence level of 95%. The final totals are:

$$S = -14 \text{ nm}$$
$$S_L = +1.15 \times 10^{-8} L$$
$$R = \pm 28 \text{ nm}$$
$$R_L = \pm 6.22 \times 10^{-8} L$$

where L is the length of the bar in metres.

Thus a full uncertainty statement for the interferometer is

The central length measurement uncertainty for the Primary Length Bar Interferometer is -14 nm $\pm$ 28 nm $+ 1.15 \times 10^{-8} L \pm 6.22 \times 10^{-8} L$ at a confidence level of 95%, for a bar of length L metres.

Depending on how the errors are combined, it is possible to obtain different estimates of the error for a particular measured length.

Firstly, the maximum and minimum possible values can be calculated as per the guidelines: $(S + S_L + R + R_L)$ and $(S + S_L - R - R_L)$ respectively. This will be referred to as the RECOMMENDED uncertainty estimate.

Secondly, the quadrature sum of the random uncertainties can be either added or subtracted from the systematic error total: $(S + S_L \pm \sqrt{R^2 + R_L^2})$. This will be referred to as the STANDARD uncertainty estimate.

The final method of combining the errors is that recommend by the BIPM where the systematic errors are combined in quadrature with the random errors to produce two figures, one length dependent, the other length independent, which are then added in quadrature: $S^2 + S_L^2 + R^2 + R_L^2$. This results in a figure of ± 30 nm ± 64 *L* nm. For comparison, the NPL Length Bar Machine has an uncertainty of length measurement of ± 68 ± 350 *L* nm). This will be referred to as the BIPM uncertainty estimate (this is the most common technique of quoting uncertainties for metrological purposes).

These different combinations are plotted in figure 10.2.

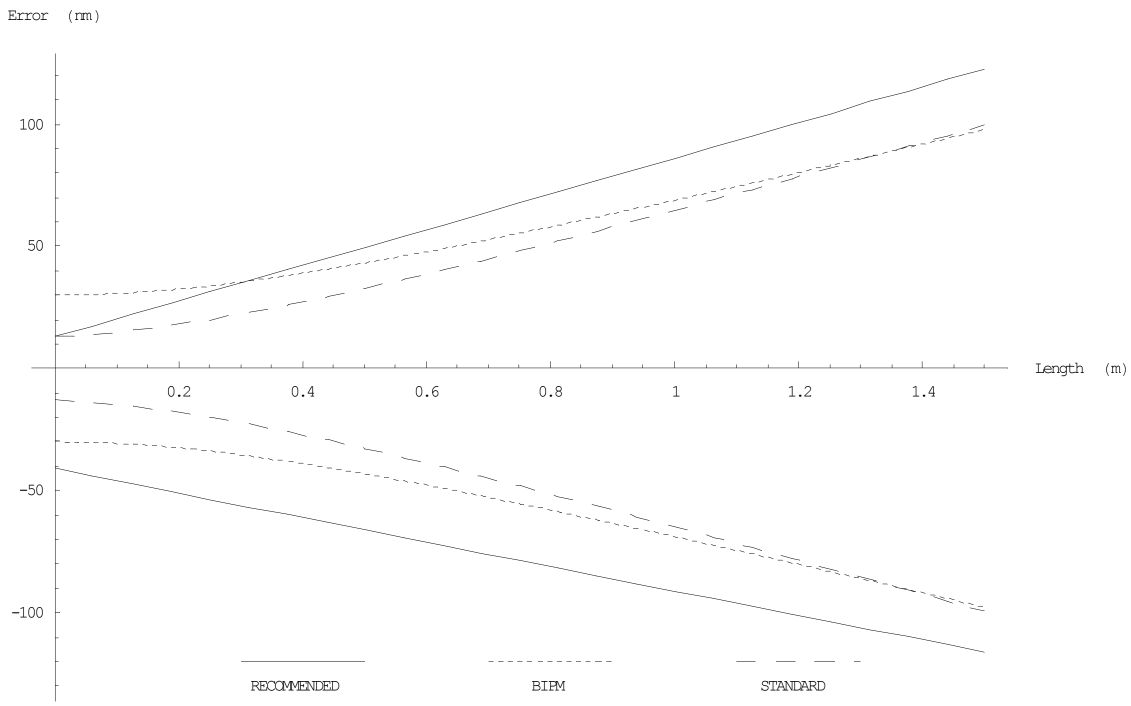


Figure 10.2 *- Plot of total uncertainty in length measurement over length range 0.1 - 1.5 m*

The differences between the techniques are due to the whether they sum the components in quadrature (sign symmetric) or arithmetically (sign asymmetric).The difference between the three techniques is approximately 20 nm, though this depends on the length of the bar. Except for bars of length 300 mm and below, the RECOMMENDED uncertainty is larger than the other techniques and is thus more 'safe' to quote if a simple analysis is required. The BIPM and STANDARD estimates are in good agreement for longer bars. Thus the importance of quoting the result in the most comprehensive form, where all the terms are listed, can be seen.

10.6 POSSIBLE STEPS TO IMPROVE THE ACCURACY

This accuracy can be improved significantly by reducing the uncertainty associated with the thermal expansion coefficient of the bar. As detailed in chapter 8, the interferometer was also designed to measure the coefficient of linear thermal expansion of length bars (and long gauge blocks). The contribution of the uncertainty in thermal expansion coefficient is $\pm 1.5 \times 10^{-8} L$ for an uncertainty of $\pm 0.5 \times 10^{-6}$ K^{-1} in α. From § 8.6 it is seen that by measuring the expansion coefficient in the interferometer, this can be reduced to an uncertainty of between ± 0.2 and $\pm 0.05 \times 10^{-6}$ K^{-1}, which corresponds to a length measurement uncertainty of between $\pm 6 \times 10^{-9} L$ and $\pm 1.5 \times 10^{-9} L$.

10.7 COMBINED UNCERTAINTY BUDGETS OF INSTRUMENTS

As stated in § 9.4 the differences between the measurements of set 1455 in the LBM and the LBI all fall within the uncertainty budget of the LBM alone, except for the 300 mm bar which has been explained. It was thus not necessary to consider the combination of the uncertainty budgets of the two instruments. For reference, this will now be discussed briefly. When comparing two results from different instruments it should be remembered that the results are given as single values with confidence limits. To a good approximation, the errors of the two instruments are randomly distributed and can be combined statistically. Standard statistical tests [**11**] can be used to ascertain a confidence level for whether or not the two sets of results share a common overlap of any statistical significance. In the case of the results given in chapter 9, the differences between the two instruments' results are all within the 95% confidence limits of the LBM uncertainty budget alone, and so there is 95% confidence that the results agree, within the stated uncertainties of the instruments.

[2002 re-release note: Since the thesis was completed, the *Guide to the Expression of Uncertainty in Measurement* (*GUM*), published by ISO, has become the *de facto* standard for uncertainty budget preparation. The style set out in the GUM is quite different to that presented in this thesis.]

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